Lattice Boltzmann Methods In Acoustics

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Abstract

This poster considers the application of the lattice Boltzmann model to simulate acoustical phenomena. The lattice Boltzmann model is described and the validity and limitations of the technique applied to sound fields is discussed. Simulation results are presented and compared with theory for plane wave propagation in a pipe and in an unconfined medium and for acoustic streaming between two parallel walls.

Intoduction

The lattice Boltzmann model (LBM) is a relatively recent development in the simulation of isothermal, incompressible fluid flows [1]. The technique has developed from the lattice gas model (LGM) which evolves idealized 'particles' on a regular hexagonal grid [2]. The LBM is also a grid based model, although different shaped grids can be used, however it no longer tracks individual particles but considers a simplified kinetic equation which incorporates the underlying properties of the fluids and which satisfies the macroscopic fluid equations. The LMB can be seen to be a discrete realization of the classical Boltzmann equation. All acoustical phenomena incorporate, by definition, a density variation, however, provided this is small, with respect to the mean density, an incompressible technique, such as the LBM, can be applied. Here we apply the LMB in this incompressible limit.

The Lattice Boltzmann Model

We consider a two-dimensional lattice Boltzmann model on a hexagonal grid with six links in the direction of the unit vectors $\boldsymbol{e}_i = \cos(2\pi i/6)\boldsymbol{i} + \sin(2\pi i/6)\boldsymbol{j}$ for $i = 1, 2, ..., 6$ where $\boldsymbol{i}$ and $\boldsymbol{j}$ are orthogonal unit vectors. The fluid at position $\boldsymbol{r}$ and time t is described in terms of the distribution functions $f_i(\boldsymbol{r}, t)$, $i = 0, 1, ..., 6$ along the six directions of the lattice and at rest at the site ($i = 0$). The physical parameters of interest, the fluid density ρ and velocity $\boldsymbol{u}$, can be found from the distribution functions as

$$\rho = \sum_i f_i \quad \text{and} \quad \rho\boldsymbol{u} = \sum_i f_i\boldsymbol{e}_i, \tag{1}$$

where $\boldsymbol{e}_0$ is the null vector. The distribution functions evolve according to the lattice Boltzmann equation

$$f_i(\boldsymbol{r}+\boldsymbol{e}_i, t+1) - f_i(\boldsymbol{r}, t) = -\frac{1}{\tau}\left[f_i(\boldsymbol{r},t) - \overline{f}_i(\boldsymbol{r},t)\right], \tag{2}$$

where $\tau > 0.5$ is a free variable and $\overline{f}_i$ is the equilibrium distribution function given by

$$\overline{f}_i(\boldsymbol{r},t) = \begin{cases} \rho(d_0 - u^2) & \text{for } i = 0 \\ \rho\left[\frac{(1-d_0)}{6} + \frac{2}{6}\boldsymbol{e}_i\cdot\boldsymbol{u} + \frac{2}{3}(\boldsymbol{e}_i\cdot\boldsymbol{u})^2 - \frac{u^2}{6}\right] & \text{for } i = 1,..,6. \end{cases} \tag{3}$$

Thus given an initial value for $f_i(\boldsymbol{r}, t_0)$ (which can which can be taken as the equilibrium value calculated using equation (3)) the value of f_i, and hence the macroscopic parameters, can be found at each site at subsequent times.
With this choice of equilibrium distribution the macroscopic equations satisfy the following equations up to second order in the velocities:

$$\partial_t\rho + \partial_\alpha\rho u_\alpha = 0 \tag{4}$$

and

$$\partial_t\rho u_\alpha + \partial_\beta\rho u_\beta u_\alpha = -\partial_\alpha\left[\frac{\rho(1-d_0)}{2}c^2\right] + \nu\partial_\beta\partial_\beta\rho u_\alpha + \partial_\alpha\zeta\partial_\beta\rho u_\beta, \tag{5}$$

where $\nu = (\tau - 1/2)/4$ and $\zeta = (\tau - 1/2)[1/4 - (1-d_0)/2]$ are the kinematic and bulk viscosities and Greek subscript have been used to label the vector components. These equations are the continuity and Navier-Stokes equation for an incompressible fluid. In this poster we restrict the density variation of the sound waves to be less than 1 % of the mean density thus insuring that we remain within the incompressible limit. The other limit to the range of situations where the LBM can be applied is that the scheme is accurate up to second order in u/c_s where c_s is the speed of sound. Thus we are also restricted to use velocities which are small compared to the speed of sound in the model. Here we insure that $u/c_s < 10$. The pressure term in equation (5) is $p = (1-d_0)/2$ which gives the speed of sound as

$$c_s = \sqrt{\frac{(1-d_0)}{2}}. \tag{6}$$

Three different boundary conditions are applied at the edges of the grid: continuous or periodic boundary conditions which wrap the simulation round onto the opposite edge of the grid giving the impression of a larger grid on which the fluid flow is repeated periodically; no-slip boundary conditions which specify zero velocity at a solid boundary [3]; and wind-tunnel boundary conditions which specify the density and velocity of the fluid at the boundary by specifying the value of f_i at the boundary, here we use $\tau = 1$ so the required values of f_i are simply their equilibrium values $\overline{f}_i$. The parameter d_0 was set to 1/2 in all the simulations.

Simulation of Sound Waves

Standing waves were simulated on a grid with n sites in the x-direction and m sites in the y-direction. A wave with wavelength n lattice units was then set up by initializing the density and velocity according to linear wave theory. The simulation was then allowed to evolve and the density and velocity measured at the center of the grid at subsequent times. This was done with no-slip boundary conditions at $y = 0$ and $y = m$ to simulate a sound wave in a pipe; and with continuous boundary conditions at $y = 0$ and $y = m$ to simulate an unconfined wave. In both cases continuous boundary conditions were applied at $x = 0$ and $x = \lambda$. This simulates a standing wave between two perfect reflectors when no energy is added to the wave and it decays freely. The density variation is shown in figure 1 for a wave in a pipe and in figure 2 for an unconfined wave both for $m = 68$ and $n = 135$, also shown is the theoretical damping rate. All the values are in lattice units which can be compared to actual waves by considering the values of the dimensionless variables. The action of the walls is seen to have a significant affect on damping the wave in a pipe and in both cases the damping rate is in good agreement with theory. This has been observed for a wide range of the model parameters [4].

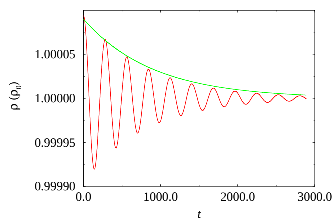

Figure 1. ——— The density variation, measured in units of the mean density, as a function of time for a standing wave in a pipe. ——— The theoretical damping rate of the wave.

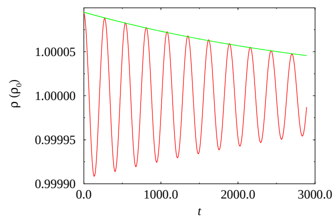

Figure 2. ——— The density variation, measured in units of the mean density, as a function of time for an unconfined wave. ——— The theoretical damping rate of the wave.

Acoustic Streaming

Acoustic streaming is a steady state fluid motion caused by the attenuation of a sound wave in a viscous fluid. The streaming can be set up by the attenuation of a sound wave either due to its interaction with the fluid or its interaction with a boundary. Here we consider the later, and in particular, acoustic streaming produced by the attenuation of a standing sound wave in a pipe. This manifests itself as a second order velocity with time averaged components, for $0 < y < y_1$, given by [5]

$$\langle u_2\rangle_t = -\frac{U^2\sin(2kx)}{8\beta c}\left\{\beta e^{-\beta y}\left[4\sin(\beta y) + 2\cos(\beta y) + e^{-\beta y}\right] + \frac{3}{4}\left[\frac{2\beta y_1 - 7}{y_1}\right] - \frac{3}{4}\left[\frac{6\beta y_1 - 7}{y_1^3}\right]z^2\right\} \tag{7}$$

and

$$\langle v_2\rangle_t = -\frac{2kU^2\cos(2kx)}{8\beta c}\left\{e^{-\beta y}\left[4\sin(\beta y) + 3\cos(\beta y) + \frac{1}{2}e^{-\beta y}\right] + \frac{3}{4}\left[\frac{2\beta y_1 - 7}{y_1}\right]z - \frac{1}{4}\left[\frac{6\beta y_1 - 7}{y_1^3}\right]z^3\right\}, \tag{8}$$

where $< ... >_t$ represents a time average over one period of the sound wave, $1/\beta = (2\nu/\omega\rho)^{1/2}$ is a measure of the distance from the boundary to which the viscosity effects the fluid, U is the amplitude of the velocity oscillations, y_1 is the position of the center of the pipe, c is the speed of sound and $z = (y_1 - y)$. When a linear sound wave is simulated in a tube and the velocity is averaged over a period the fist order velocity (the sinusoidal velocity variation of the wave) averages to zero and the result is the time averaged velocity components $\langle u_2\rangle_t$ and $\langle v_2\rangle_t$. The LGM has previously been applied to study acoustic streaming in a pipe [6]. The results compared well with theory despite the inherent noise of the LGA which necessitated a restrictively large amount of averaging to observe the second order velocities, and the limited range of Reynolds numbers which could be obtained. The LMB simulations are noise-free and a much larger Reynolds number can be achieved. The simulation was performed on a tube of length λ with windtunnel boundary conditions applied at $x = 0$ and $x = \lambda$. The results of the simulation are shown in figure 3 and the theoretical prediction, equations (7) and (8), is shown in figure 4 for comparison. The same basic vortex is seen in both plots and a smaller vortex, close to the boundary at $y = 0$ is also observed in the simulation results. In a pipe a sound wave is affected by the proximity of the walls. This is observed by an attenuation of the wave, as was seen in figure 1 and also by a small change in the speed with which the sound propagates. Since the lattice Boltzmann model is discretised in space and time it is not possible to choose an integer value for the wavelength and pipe thickness for which the wave period is also an integer. Thus the averaging process will never average the first order (sinusoidal) velocity to exactly zero. This small error influences the results for the second order velocity. The wind tunnel boundary conditions apply a constant velocity across the pipe at the edges without taking into account the boundary layer. Both of these facts influence the discrepancy between the simulated results and the theoretical predictions.

Acknowledgements

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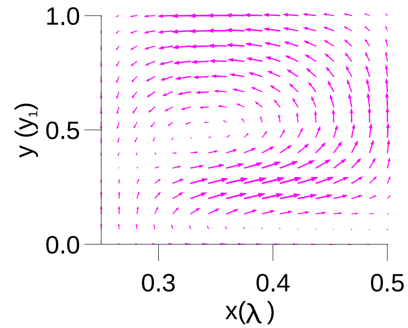

Figure 3. A simulated acoustic streaming cell for $\lambda/4 < x < \lambda/2$ and $0 < y < y_1$.

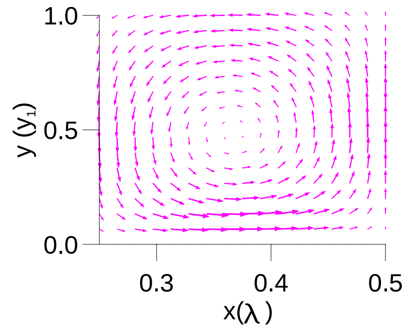

Figure 4. A theoretical acoustic streaming cell for $\lambda/4 < x < \lambda/2$ and $0 < y < y_1$.

Conclusion

The lattice Boltzmann model has been applied to simulate sound propagation in the incompressible limit where the density variation is small with respect to the mean density. Propagation of unconfined waves and waves in a pipe have been simulated and seen to show good agreement with theory suggesting that the lattice Boltzmann approach is valid. Acoustic streaming produced by the attenuation of a forced standing wave in a pipe has also been simulated, the results showing the same features as the theory.

References

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